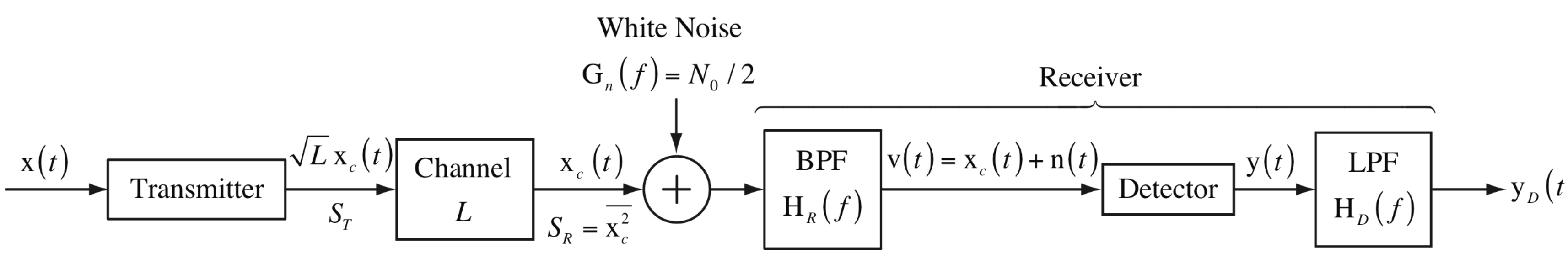
Noise in Analog Modulation Systems

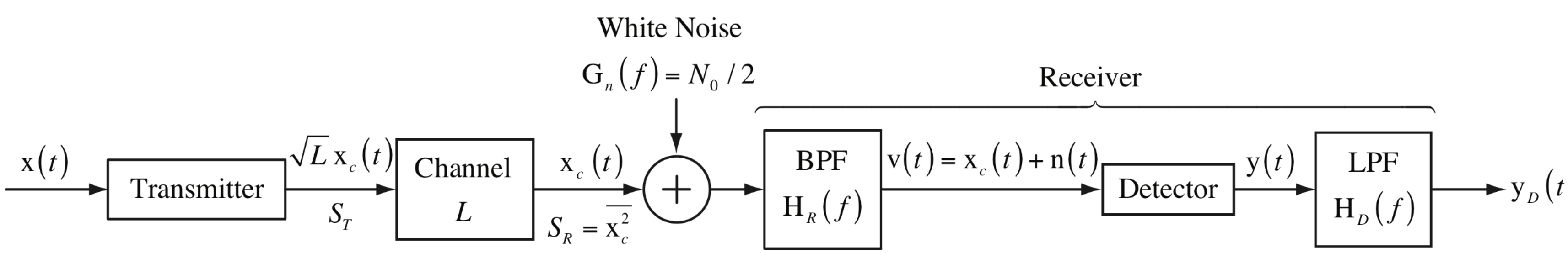
Bandpass Noise

Referring to the communication system diagram below, $\mathrm{x}(t)$ is the message, with bandwidth W, normalized such that $|\mathrm{x}(t)| \le 1$, $S_{\mathrm{x}} = \overline{\mathrm{x}^2}$ and $\langle \mathrm{x}^2(t) \rangle \le 1$. After transmission it experiences loss L in the channel and arrives at the receiver as $\mathrm{x}_c(t)$ with signal power S_R. So before the channel loss it is $\sqrt{L}\,\mathrm{x}_c(t)$ with signal power S_T. The channel is assumed essentially distortionless with negligible time delay.



Bandpass Noise

At the receiver input, noise is injected. This noise represents all the equivalent noise effects of all noise sources referred to the receiver input. All noise sources are assumed to be white and uncorrelated with the signal or any other noise source. The predetection filter is a bandpass filter with frequency response $\mathrm{H}_R(f)$ having unity gain over the transmission bandwidth B_T. Then the total signal plus noise at the detector is $\mathrm{v}(t) = \mathrm{x}_c(t) + \mathrm{n}(t)$ where $\mathrm{n}(t)$ is the **predetection noise**. Then the detected signal is lowpass filtered to yield the final signal $\mathrm{y}_D(t)$.

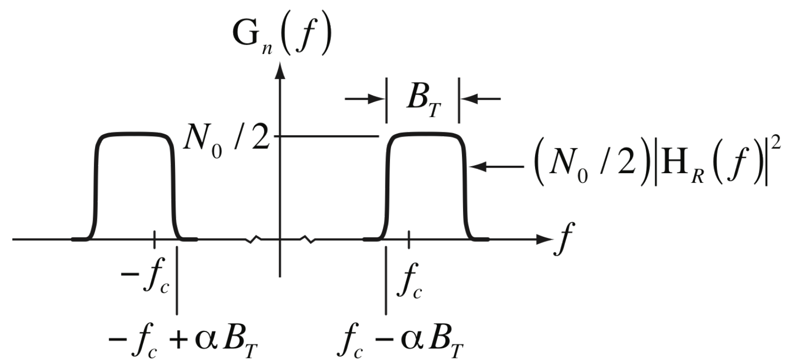


Bandpass Noise

The filtered noise at the output of the bandpass filter has the power spectrum $G_n(f) = (N_0/2)|H_R(f)|^2$. The bandpass filter is assumed to have a nearly rectangular shape with bandwidth B_T. Therefore the noise power from the bandpass filter is $N_R = \int_{-\infty}^{\infty} G_n(f)\,df = N_0 B_T$.

The predetection signal-to-noise ratio is defined as $(S/N)_R \triangleq \dfrac{S_R}{N_R} = \dfrac{S_R}{N_0 B_T}$.

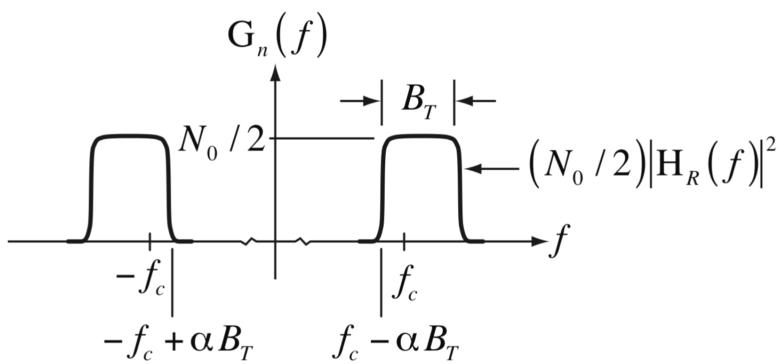
$$G(f) = \frac{N_0}{2} \longrightarrow \boxed{H_R(f)} \longrightarrow G_n(f) = \frac{N_0}{2}|H_R(f)|^2$$



Bandpass Noise

Define $\gamma \triangleq \dfrac{S_R}{N_0 W}$. Then $(S/N)_R = \dfrac{W}{B_T}\gamma$ and $(S/N)_R \le \gamma$ because $B_T \ge W$. Notice that f_c is not drawn at the center of the passband. The lower cutoff frequency is indicated as $f_c - \alpha B_T$. If $\alpha = 1/2$ we have DSB. If $\alpha = 0$ we have USSB. If $\alpha = 1$ we have LSSB.

$$G(f) = \frac{N_0}{2} \longrightarrow \boxed{H_R(f)} \longrightarrow G_n(f) = \frac{N_0}{2}\left|H_R(f)\right|^2$$

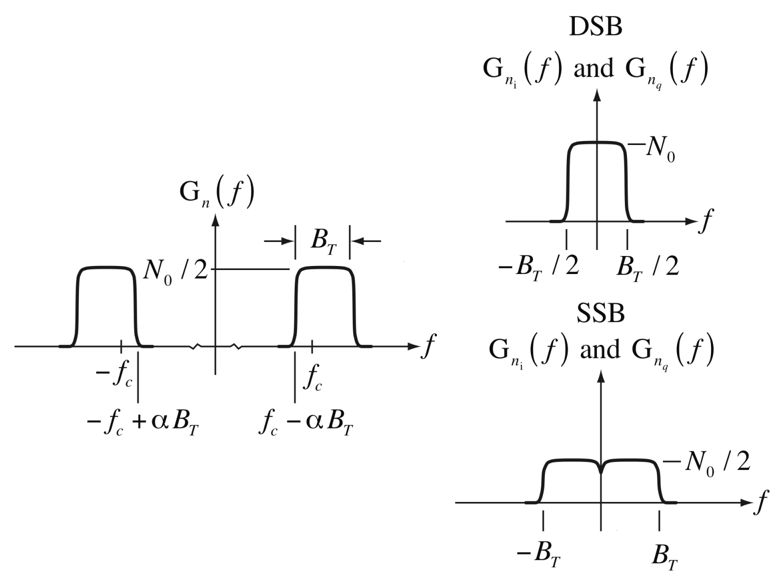


Bandpass Noise

Let $\mathrm{n}(t)$ be a sample function from an **additive white gaussian noise** (**AWGN**) process. Then $\overline{\mathrm{n}} = 0$ and $\overline{\mathrm{n}^2} = \sigma_N^2 = N_R$. We can express $\mathrm{n}(t)$ in the form $\mathrm{n}(t) = \mathrm{n_i}(t)\cos(\omega_c t) \;-\; \mathrm{n}_q(t)\sin(\omega_c t)$ where $\mathrm{n_i}(t)$ is the in-phase component and $\mathrm{n}_q(t)$ is the quadrature component. They are both stationary and gaussian and $\overline{\mathrm{n_i}} = \overline{\mathrm{n}_q} = 0$, $\overline{\mathrm{n_i}(t)\mathrm{n}_q(t)} = 0$ and $\overline{\mathrm{n_i^2}} = \overline{\mathrm{n}_q^2} = \overline{\mathrm{n}^2} = N_R$.

The power spectral densities of the quadrature components are

$$\mathrm{G}_{\mathrm{n_i}}(f) = \mathrm{G}_{\mathrm{n}_q}(f) = \begin{bmatrix} \mathrm{G}_n(f+f_c)\mathrm{u}(f+f_c) \\ +\mathrm{G}_n(f-f_c)\mathrm{u}(f_c-f) \end{bmatrix}$$

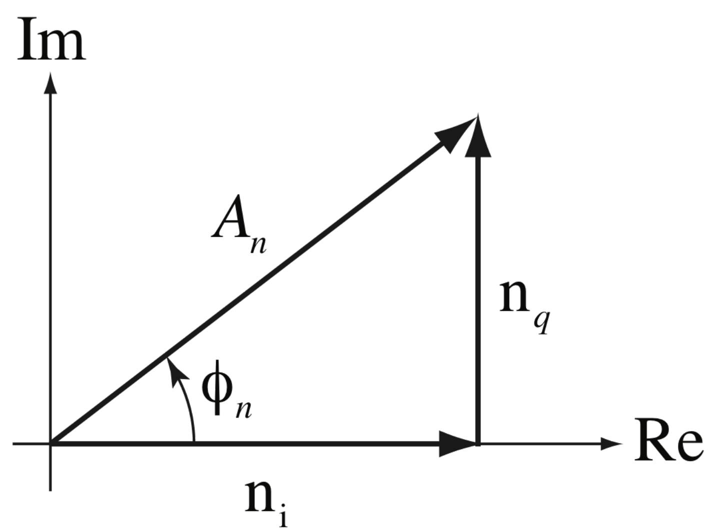


Bandpass Noise

Bandpass noise can also be expressed in envelope-and-phase form as $\mathrm{n}(t) = A_n(t)\cos(\omega_c t + \phi_n(t))$ in which $A_n(t)$ is the envelope and $\phi_n(t)$ is the phase. These are related to the quadrature components as illustrated below.

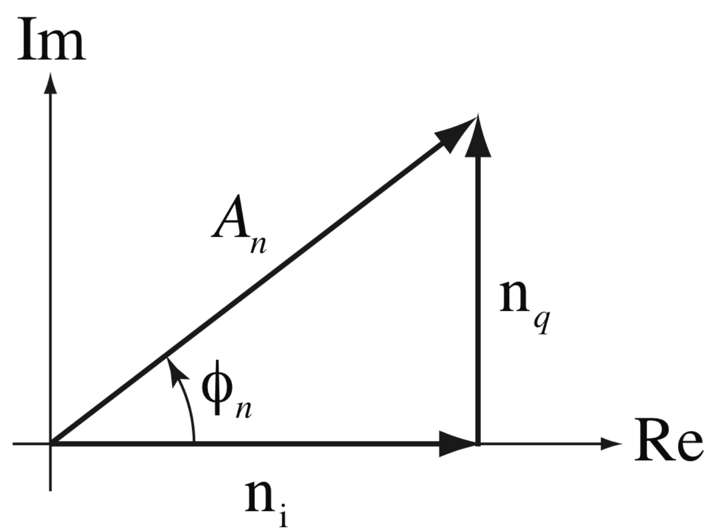
$A_n^2 = \mathrm{n}_\mathrm{i}^2 + \mathrm{n}_q^2$, $\phi_n = \tan^{-1}(\mathrm{n}_q / \mathrm{n}_\mathrm{i})$, $\mathrm{n}_\mathrm{i} = A_n \cos(\phi_n)$ and $\mathrm{n}_q = A_n \sin(\phi_n)$

Because of these relationships the PDF of the envelope is **Rayleigh** distributed, $\mathrm{p}_{A_n}(A_n) = (A_n / N_R) e^{-A_n^2/2N_R}\,\mathrm{u}(A_n)$.



Bandpass Noise

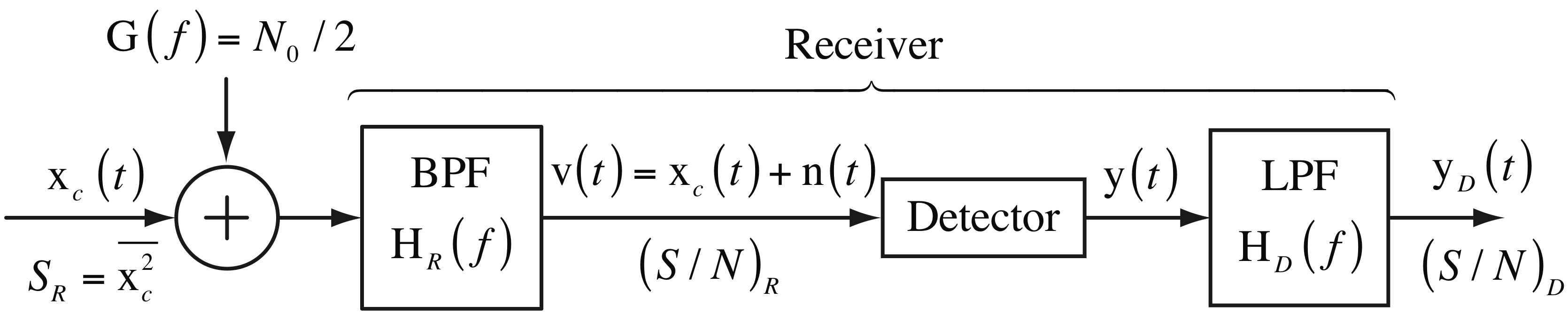
The mean value of the envelope is $\overline{A_n} = \sqrt{\pi N_R / 2}$ and its mean-squared value is $\overline{A_n^2} = 2N_R$. The phase is uniformly distributed over a range of 2π radians and is independent of the envelope.



Linear CW Modulation with Noise

Demodulation will be represented by

$$y(t) = \begin{cases} v_i(t) & , \text{ Synchronous Detector} \\ A_v(t) - \overline{A_v} & , \text{ Envelope Detector} \end{cases}$$



Linear CW Modulation with Noise

(This development is not identical to the one in the book but the final conclusions are the same.)

For DSB and synchronous detection $\mathrm{x}_c(t) = A_c\,\mathrm{x}(t)\cos(\omega_c t)$,

$$\mathrm{v}(t) = \left[A_c\,\mathrm{x}(t) + \mathrm{n}_\mathrm{i}(t)\right]\cos(\omega_c t) - \mathrm{n}_q(t)\sin(\omega_c t),$$

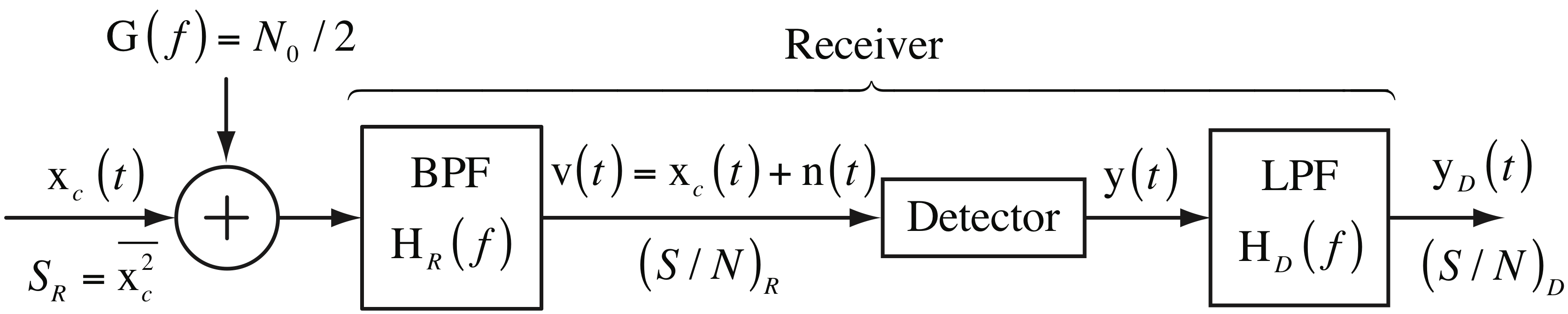
$\mathrm{y}(t) = (1/2)\mathrm{v}_\mathrm{i}(t) = (1/2)\left[A_c\,\mathrm{x}(t) + \mathrm{n}_\mathrm{i}(t)\right]$ and, after lowpass filtering with $\mathrm{H}_D(f)$,

$\mathrm{y}_D(t) = (1/2)\left[A_c\,\mathrm{x}(t) + \mathrm{n}_\mathrm{i}(t)\right]$. The noise power spectrum of $\mathrm{n}_\mathrm{i}(t)$ is $\mathrm{G}_{n_\mathrm{i}}(f) = N_0\Pi(f/2W)$,

lowpass filtered white noise. Using $N_D = \overline{(\mathrm{n}_\mathrm{i}/2)^2}$, $S_D = A_c^2\overline{\mathrm{x}^2}/4$, $S_R = \overline{\mathrm{x}_c^2} = A_c^2 S_\mathrm{x}/2$ and $B_T = 2W$,

the signal-to-noise ratio after detection and lowpass filtering is

$$(S/N)_D = \frac{(1/4)A_c^2\overline{\mathrm{x}^2}}{(1/4)\overline{\mathrm{n}_\mathrm{i}^2}} = \frac{A_c^2 S_x}{N_0 B_T} = \frac{2S_R}{N_0 B_T} = \frac{S_R}{N_0 W} = \gamma = 2(S/N)_R$$



Linear CW Modulation with Noise

(This development is not identical to the one in the book but the final conclusions are the same.)

For AM and synchronous detection (assuming $\mu = 1$) $x_c(t) = A_c[1 + x(t)]\cos(\omega_c t)$,

$v(t) = \{A_c[1 + x(t)] + n_i(t)\}\cos(\omega_c t) - n_q(t)\sin(\omega_c t)$, $y(t) = (1/2)\{A_c[1 + x(t)] + n_i(t)\}$ or, if there is a DC block, $y(t) = y_D(t) = (1/2)[A_c\, x(t) + n_i(t)]$, the same result as for DSB.

The noise power spectrum is also the same. Using $N_D = \overline{(n_i/2)^2}$,

$$S_R = \overline{x_c^2} = A_c^2(1 + S_x)/2 \Rightarrow S_x = \frac{2S_R}{A_c^2} - 1,$$

$$S_D = (1/4)A_c^2 S_x = \frac{2S_R - A_c^2}{4} = \frac{S_R}{2} - \frac{S_R}{2(1 + S_x)} = \frac{S_R S_x}{2(1 + S_x)} \text{ and } B_T = 2W,$$

the signal-to-noise ratio after detection and lowpass filtering is

$$(S/N)_D = \frac{S_D}{(1/4)\overline{n_i^2}} = \frac{\dfrac{S_R S_x}{2(1 + S_x)}}{(1/4)N_0 B_T} = \frac{S_R S_x}{N_0 W(1 + S_x)} = \frac{S_x}{1 + S_x}\gamma$$

Linear CW Modulation with Noise

(This development is not identical to the one in the book but the final conclusions are the same.)

For SSB and synchronous detection $x_c(t) = (A_c/2)\left[x(t)\cos(\omega_c t) \mp \hat{x}(t)\sin(\omega_c t)\right]$,

$\overline{x_c^2(t)} = (A_c^2/4)(1/2)\left[\underbrace{\overline{x^2(t)}}_{=S_x} + \underbrace{\overline{\hat{x}^2(t)}}_{=S_x}\right] = \dfrac{A_c^2 S_x}{4} = S_R$ and $y_D(t) = (1/2)\left[\dfrac{A_c\, x(t)}{2} + n_i(t)\right]$.

Using $N_D = \overline{(n_i/2)^2} = N_0 W/4$, $S_x = \dfrac{4S_R}{A_c^2}$, $S_D = \dfrac{A_c^2 S_x}{16} = \dfrac{S_R}{4}$ and $B_T = W$,

the signal-to-noise ratio after detection and lowpass filtering is

$$(S/N)_D = \frac{S_D}{N_D} = \frac{S_R/4}{N_0 W/4} = \frac{S_R}{N_0 W} = \gamma$$

Linear CW Modulation with Noise

For AM and envelope detection (assuming $\mu = 1$) $\mathrm{x}_c(t) = A_c[1+\mathrm{x}(t)]\cos(\omega_c t)$,

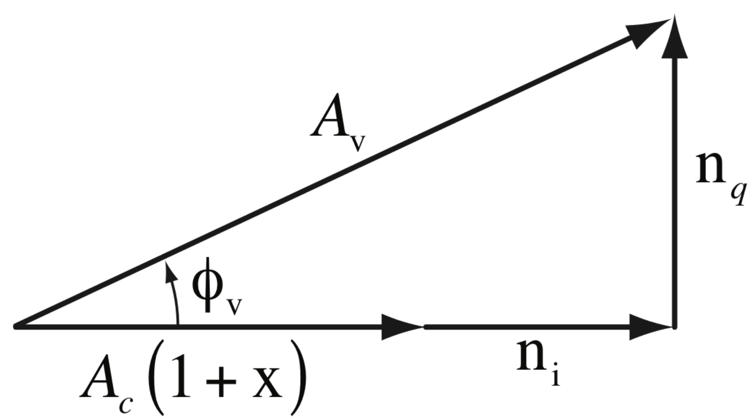
$\mathrm{v}(t) = \{A_c[1+\mathrm{x}(t)] + \mathrm{n_i}(t)\}\cos(\omega_c t) - \mathrm{n}_q(t)\sin(\omega_c t)$ and

$$\mathrm{A_v}(t) = \sqrt{\{A_c[1+\mathrm{x}(t)] + \mathrm{n_i}(t)\}^2 + \mathrm{n}_q^2(t)} \text{ and } \phi_\mathrm{v}(t) = \tan^{-1}\left(\frac{\mathrm{n}_q(t)}{A_c[1+\mathrm{x}(t)] + \mathrm{n_i}(t)}\right)$$

Case 1: Signal much larger than noise, $A_c^2 >> \overline{\mathrm{n}^2}$

$$A_\mathrm{v}(t) \cong A_c[1+\mathrm{x}(t)] + \mathrm{n_i}(t) \Rightarrow \mathrm{y}_D(t) = \mathrm{A_v}(t) - \overline{A_\mathrm{v}} \cong A_c\,\mathrm{x}(t) + \mathrm{n_i}(t)$$

This is exactly the same result obtained with synchronous detection and the signal-to-noise ratio is also the same. So, in the case of a large signal-to-noise ratio, synchronous detection and envelope detection have approximately the performance quality.



Linear CW Modulation with Noise

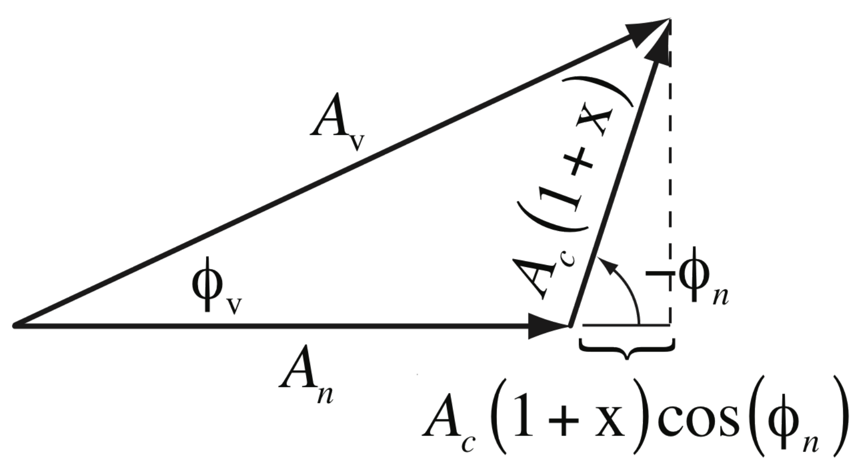
Case 2: Noise much larger than signal, $A_c^2 << \overline{\mathrm{n}^2}$

Since the noise is dominant, let it be the reference for the phasor diagram

$\mathrm{n}(t) = \mathrm{A}_n(t)\cos(\omega_c t + \phi_n(t))$ and $\mathrm{A}_\mathrm{v}(t) = \mathrm{A}_n(t) + A_c[1+\mathrm{x}(t)]\cos(\phi_n(t))$

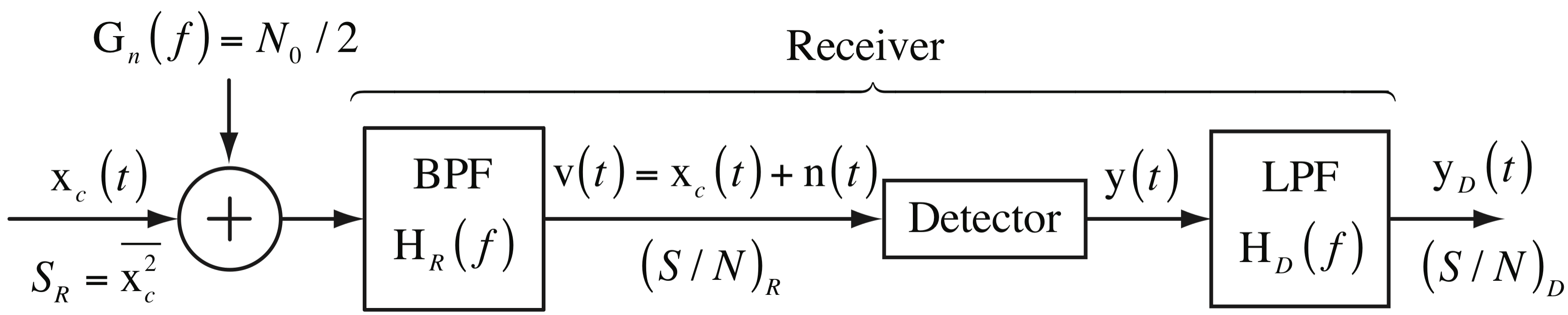
$\mathrm{y}(t) = \mathrm{A}_n(t) + A_c\,\mathrm{x}(t)\cos(\phi_n(t)) - \overline{A_n}$ and $\overline{A_n} = \sqrt{\pi N_R / 2}$

This message is multiplied by the cosine of the noise phase angle and is therefore unintelligible. In synchronous detection, with noise dominant, the signal is buried in noise but not multiplied by the cosine of the phase angle and is therefore still intact (but hard to pick out because of the dominance of the noise).



Angle CW Modulation with Noise

In the analysis of PM and FM systems, demodulation will be represented by $\mathrm{y}(t) = \begin{cases} \phi_{\mathrm{v}}(t) & , \quad \text{Phase Detector} \\ \dot{\phi}_{\mathrm{v}}(t)/2\pi & , \quad \text{Frequency Detector} \end{cases}$. The predetection part of an angle modulation receiver is the same as for amplitude modulation. The received signal is $\mathrm{x}_c(t) = A_c \cos(\omega_c t + \phi(t))$ where $\phi(t) = \phi_\Delta(t)\mathrm{x}(t)$ for PM and $\dot{\phi}(t) = 2\pi f_\Delta \mathrm{x}(t)$ for FM. The carrier has a constant amplitude, so $S_R = A_c^2/2$ and $(S/N)_R = \dfrac{A_c^2}{2N_0 B_T}$ and this is often called the **carrier-to-noise ratio (CNR)**.



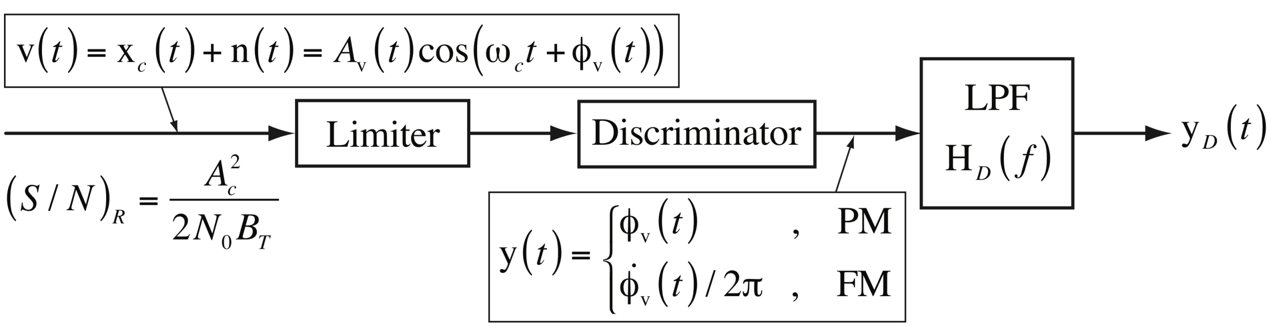
Angle CW Modulation with Noise

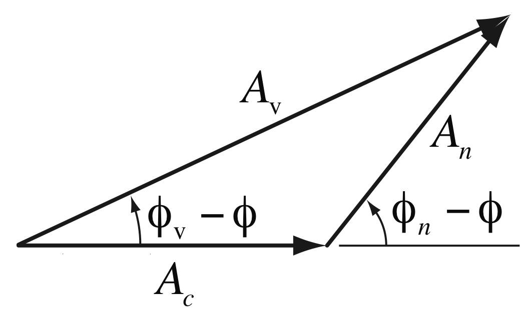
The PM and FM detector model is shown below. The limiter suppresses any amplitude variation $\mathrm{A_v}(t)$. Express the noise in envelope-and-phase form as $\mathrm{n}(t) = \mathrm{A}_n(t)\cos(\omega_c t + \phi_n(t))$. Then

$$\mathrm{v}(t) = A_c \cos(\omega_c t + \phi(t)) + \mathrm{A}_n(t)\cos(\omega_c t + \phi_n(t)).$$

Then, using the $A_c \cos(\omega_c t + \phi(t))$ phasor as the reference for angle,

$$\phi_{\mathrm{v}}(t) - \underbrace{\phi(t)}_{\substack{\text{Zero in} \\ \text{the phasor} \\ \text{diagram}}} = \tan^{-1}\left(\frac{\mathrm{A}_n(t)\sin(\phi_n(t) - \phi(t))}{A_c + \mathrm{A}_n(t)\cos(\phi_n(t) - \phi(t))}\right)$$



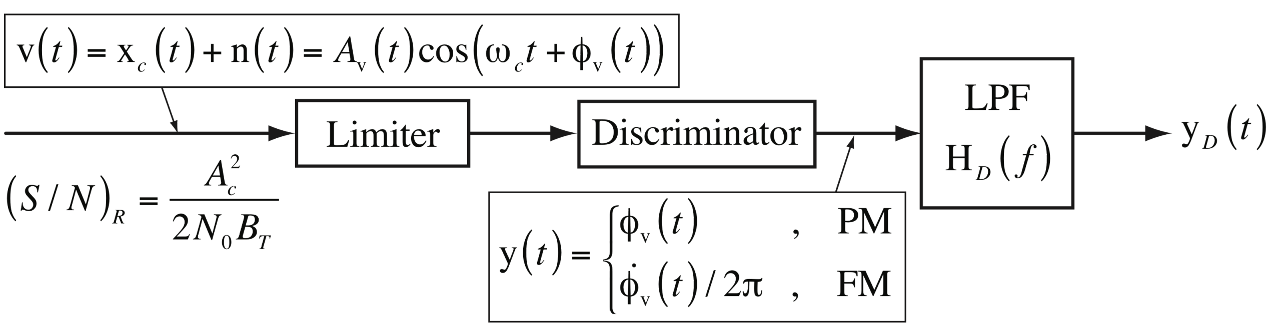


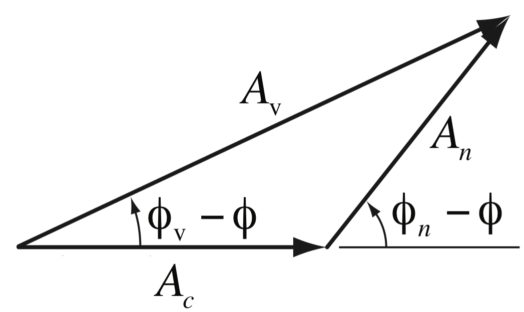
Angle CW Modulation with Noise

$$\phi_v(t) - \underbrace{\phi(t)}_{\substack{\text{Zero in} \\ \text{the phasor} \\ \text{diagram}}} = \tan^{-1}\left(\frac{A_n(t)\sin(\phi_n(t)-\phi(t))}{A_c + A_n(t)\cos(\phi_n(t)-\phi(t))}\right)$$

The term on the right side involves both noise and signal. To simplify the analysis consider the large CNR case $(S/N)_R >> 1$. Then $A_c >> A_n(t)$ and we can approximate the inverse tangent as being equal to its (small) argument. Then

$$\phi_v(t) \cong \phi(t) + \frac{A_n(t)\sin(\phi_n(t)-\phi(t))}{A_c + A_n(t)\cos(\phi_n(t)-\phi(t))} \cong \phi(t) + \frac{A_n(t)\sin(\phi_n(t)-\phi(t))}{A_c}$$





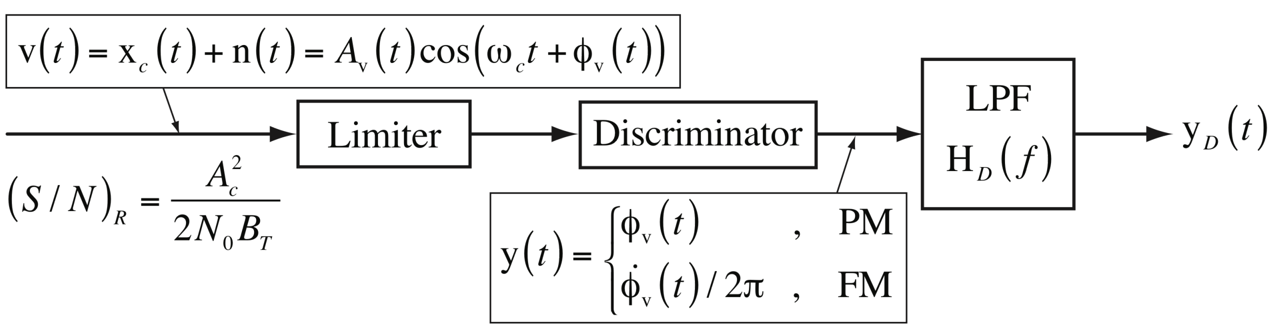
Angle CW Modulation with Noise

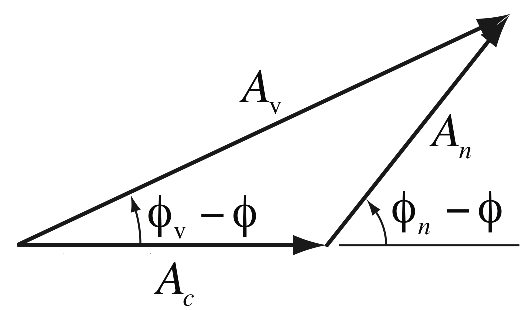
$$\phi_v(t) \cong \phi(t) + \frac{A_n(t)\sin(\phi_n(t) - \phi(t))}{A_c}$$

Next, consider the behavior of $\sin(\phi_n(t) - \phi(t))$. $\phi_n(t)$ is uniformly distributed over 2π radians. If we add to (or subtract from) $\phi_n(t)$ we don't really change the noise characteristics of $\sin(\phi_n(t) - \phi(t))$ because the range of the angle is still 2π radians. So, for purposes of signal-to-noise computation we can leave $\phi(t)$ out and have $\phi_v(t) \cong \phi(t) + \dfrac{A_n(t)\sin(\phi_n(t))}{A_c} = \phi(t) + \psi(t)$ where

$\psi(t) \triangleq \dfrac{A_n(t)\sin(\phi_n(t))}{A_c}$. Let $A_n(t)\sin(\phi_n(t)) = n_q$, the quadrature part of $n(t)$.

Then $\psi(t) = \dfrac{n_q}{A_c} = \dfrac{1}{\sqrt{2S_R}} n_q$ because $S_R = A_c^2/2$.





Angle CW Modulation with Noise

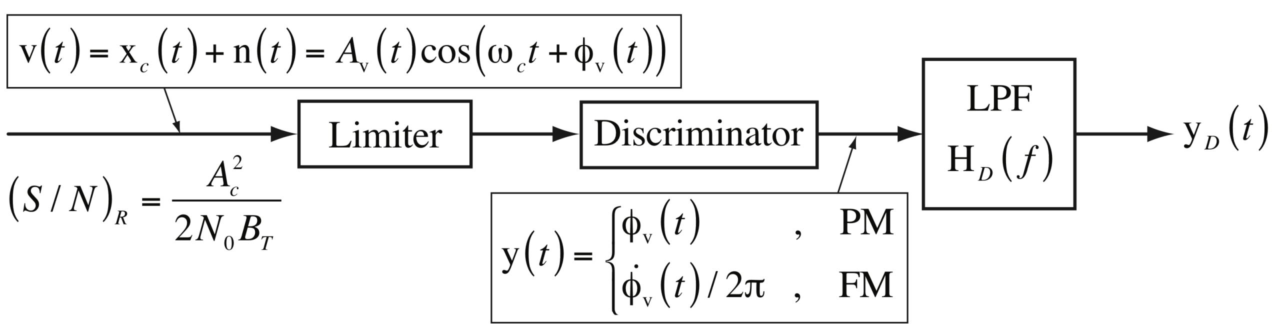
$$\phi_v(t) \cong \phi(t) + \frac{1}{\sqrt{2S_R}} n_q(t)$$

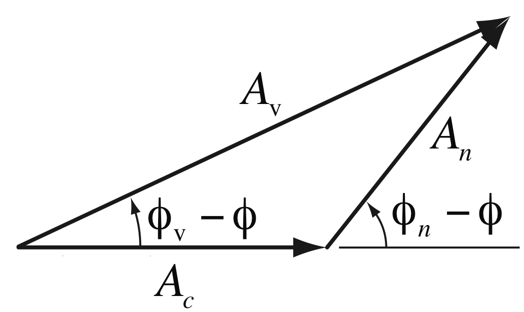
This result says that the signal phase and the phase noise are additive under high signal-to-noise ratio conditions and that the phase noise $\psi(t)$ depends on the quadrature component of $n(t)$ and decreases with increasing signal power. Now consider FM detection (without preemphasis and deemphasis).

The **instantaneous frequency noise** is $\varepsilon(t) = \dfrac{1}{2\pi}\dfrac{d}{dt}\left(\dfrac{1}{\sqrt{2S_R}} n_q(t)\right) = \dfrac{1}{2\pi}\dfrac{1}{\sqrt{2S_R}}\dot{n}_q(t)$.

$G_{n_q}(f)$ is the power spectral density of $n_q(t)$. Then the power spectral density of $\dot{n}_q(t)$ is $(2\pi f)^2 G_{n_q}(f)$ and

$$G_\varepsilon(f) = (2\pi f)^2 \left[\frac{1}{2\pi}\frac{1}{\sqrt{2S_R}}\right]^2 G_{n_q}(f) = \frac{f^2}{S_R} G_{n_q}(f) = \frac{f^2}{S_R}\frac{N_0}{2}\Pi(f/B_T)$$





Angle CW Modulation with Noise

$$G_\varepsilon(f) = \frac{f^2}{S_R}\frac{N_0}{2}\Pi(f/B_T)$$

Assuming the postdetection filter is a simple LPF with bandwidth $W < B_T$,

$$N_D = \int_{-W}^{W} \frac{f^2}{S_R}\frac{N_0}{2}\Pi(f/B_T)\,df = 2\times\frac{N_0}{S_R}\int_0^W f^2\,df = \frac{N_0}{S_R}\left[f^3/3\right]_0^W = \frac{N_0 W^3}{3S_R}$$

The destination signal power is $S_D = f_\Delta^2 S_x$. Therefore $(S/N)_D = \dfrac{f_\Delta^2 S_x}{N_0 W^3/3S_R} = 3\left(\dfrac{f_\Delta}{W}\right)^2 \dfrac{S_x S_R}{N_0 W}$

Since f_Δ/W is the deviation ratio D, $(S/N)_D = 3D^2\dfrac{S_x S_R}{N_0 W} = 3D^2 S_x\gamma$. This important result indicates that, at least in high signal-to-noise situations, one can increase the signal-to-noise ratio by increasing D *without increasing the signal power of the transmitted signal*. But we don't get something for nothing. Increasing D increases the transmitted bandwidth.